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NOTE

Removing Small Features from Computational Domains

1. INTRODUCTION

Small size features cause difficulties in the numerical solution of partial differential equations by the finite element or finite difference methods. This is because such features require refined meshes in their neighborhoods, with their attendant problems. We have devised a method to avoid these problems. It is to exclude each small feature from the computational domain by surrounding it with an artificial boundary upon which a new boundary condition is imposed. This new boundary condition is chosen to account exactly for the excluded feature. The problem in the reduced domain, which we call the reduced problem, is more convenient than is the original problem for numerical solution. The artificial boundary can be made large enough to be compatible with the original mesh, so that no refinement is needed.

This method is related to that of converting infinite domains to finite computational domains by the introduction of an artificial external boundary. In that case, an exact non-reflecting boundary condition can be imposed on the solution at the artificial external boundary [1]. In the present method artificial internal boundaries are introduced, as was done in [2] to eliminate large homogeneous regions and in [3] to eliminate re-entrant corners from computational domains. The boundary condition on the artificial boundary surrounding an excluded small feature involves the scattering matrix S for that feature.

We shall explain the method by applying it to the Helmholtz or reduced wave equation in a two-dimensional domain containing a small scatterer. We shall also show how the new boundary condition simplifies in special cases. Finally we shall prove that the method does yield the solution of the original problem.

The method is applicable to a small feature on or near a boundary, as the examples in [3] show. However, unless the boundary and boundary condition near this feature are simple, the determination of the new boundary condition may be difficult.

2. DERIVATION OF THE BOUNDARY CONDITION

We consider a solution $u(r, \theta)$ of the Helmholtz equation in a domain Ω/D of the r, θ plane:

$$\Delta u + k^2 u = 0 \qquad (r, \theta) \in \Omega/D. \tag{2.1}$$

The domain Ω contains a small subdomain D which we call an obstacle or scatterer. On ∂D, the boundary of D, we suppose that u satisfies some linear homogeneous boundary condition

$$Bu = 0 \qquad (r, \theta) \in \partial D. \tag{2.2a}$$

On $\partial\Omega$ we require that for some given operator B_1 and some given function g,

$$B_1 u = g \qquad (r, \theta) \in \partial\Omega. \tag{2.2b}$$

With the origin $r = 0$ in D, we choose a radius R large enough that D lies inside the circle $r = R$, which must be contained in Ω. Our goal is to find an operator M such that u satisfies the following boundary condition on the circle $r = R$:

$$u_r(R, \theta) = Mu(R, \theta). \tag{2.3}$$

The operator M may depend upon k, R, D, and the boundary operator B in (2.2a). However, it must be independent of u.

To find M we write

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} u_n(r)\, e^{in\theta}. \tag{2.4}$$

This representation holds in the annulus $r_0 < r \leqslant R$, where $r = r_0$ is the smallest circle enclosing D. There the Fourier coefficient $u_n(r)$ is given by

$$u_n(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta)\, e^{-in\theta}\, d\theta. \tag{2.5}$$

Separation of variables in (2.1) shows that $u_n(r)$ is a linear combination of the Bessel function $J_n(kr)$ and the Hankel function $H_n^{(1)}(kr)$:

$$u_n(r) = a_n J_n(kr) + b_n H_n^{(1)}(kr). \tag{2.6}$$

When (2.6) is used in (2.4) the series splits into a part containing Bessel functions, which is regular at the origin, and

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a part containing Hankel functions, which is outgoing at infinity. The first part can be viewed as a wave incident upon the scatterer D and the second part, as the corresponding scattered wave. As a consequence, each coefficient b_n in the scattered wave is a linear combination of the coefficients a_j in the incident wave, with weights S_{nj} which are elements of the scattering matrix S:

$$b_n = \sum_{j=-\infty}^{\infty} S_{nj} a_j. \tag{2.7}$$

Assuming that the S_{nj} are known, we now determine the a_n and b_n in terms of the u_j. First we set $r = R$ in (2.6) and use (2.7) to obtain

$$u_n(R) = J_n(kR)\, a_n + H_n^{(1)}(kR) \sum_j S_{nj} a_j. \tag{2.8}$$

This is a system of linear equations for the a_n. We denote the coefficient matrix on the right side of (2.8) by T with matrix elements

$$T_{nj} = J_n(kR)\, \delta_{nj} + H_n^{(1)}(kR)\, S_{nj}. \tag{2.9}$$

We write the elements of the inverse matrix T^{-1} as $(T^{-1})_{nj}$ and we assume that T^{-1} exists. Then we can write the solution of (2.8) as

$$a_n = \sum_j (T^{-1})_{nj}\, u_j(R). \tag{2.10}$$

Now (2.6) can be solved for b_n. With $r = R$ and with a_n given by (2.10), the result is

$$\begin{aligned} b_n &= [u_n(R) - J_n(kR)\, a_n]/H_n^{(1)}(kR) \\ &= \left[u_n(R) - J_n(kR) \sum_j (T^{-1})_{nj}\, u_j(R) \right] \Big/ H_n^{(1)}(kR). \end{aligned} \tag{2.11}$$

Next we differentiate (2.6) with respect to r, set $r = R$, and use (2.10) and (2.11) to obtain

$$\begin{aligned} \partial_r u_n(R) &= \frac{k H_n^{(1)\prime}(kR)}{H_n^{(1)}(kR)}\, u_n(R) \\ &\quad + \left\{ k J_n'(kR) - \frac{k J_n(kR)\, H_n^{(1)\prime}(kR)}{H_n^{(1)}(kR)} \right\} \\ &\quad \times \sum_j (T^{-1})_{nj}\, u_j(R). \end{aligned} \tag{2.12}$$

Finally we multiply (2.12) by $e^{in\theta}$ and sum over n. Then the left side becomes $\partial_r u(R, \theta)$. On the right side we use (2.5) to eliminate the $u_n(R)$ and then we can write (2.12) in the form

$$\begin{aligned} \partial_r u(R, \theta) &= \frac{k}{2\pi} \sum_{n=-\infty}^{\infty} \left[\frac{H_n^{(1)\prime}(kR)}{H_n^{(1)}(kR)} \int_0^{2\pi} e^{-in\theta'} u(R, \theta')\, d\theta' \right. \\ &\quad \times \left\{ J_n'(kR) - \frac{J_n(kR)\, H_n^{(1)\prime}(kR)}{H_n^{(1)}(kR)} \right\} \\ &\quad \left. \times \sum_{j=-\infty}^{\infty} (T^{-1})_{nj} \int_0^{2\pi} e^{-ij\theta'} u(R, \theta')\, d\theta' \right] e^{in\theta}. \end{aligned} \tag{2.13}$$

This result (2.13) is the desired boundary condition. It is of the form (2.3) with M a non-local integral operator defined by the right side of (2.13). The expression in braces in (2.13) can be simplified by using the Wronskian of J_n and $H_n^{(1)}$.

By using (2.13) we can formulate a problem, which we call the reduced problem, in the reduced domain Ω/R, i.e., in Ω with the disk R deleted. Here R is the disk $r < R$. It consists in solving (2.1) in Ω/R, (2.2b) on $\partial\Omega$, and (2.13) on $r = R$.

3. EXAMPLES

We shall now show how (2.13) simplifies in three special cases. First, if there is no scatterer or object contained within the circle $r = R$, then $S_{nj} = 0$ for all n and j. Then (2.9) shows that $T_{nj} = J_n(kR)\, \delta_{nj}$ so T is diagonal. It is invertible if and only if $J_n(kR) \neq 0$ for all n, i.e., if and only if k is not an eigenvalue of (2.1) in $r \leqslant R$ with the boundary condition $u(R, \theta) = 0$. When k is not an eigenvalue, $(T^{-1})_{nj} = \delta_{nj}/J_n(kR)$ and (2.13) reduces to

$$\partial_r u(R, \theta) = \frac{k}{2\pi} \sum_{n=-\infty}^{\infty} \frac{J_n'(kR)}{J_n(kR)} \int_0^{2\pi} e^{-in\theta'} u(R, \theta') \cdot e^{in\theta}. \tag{3.1}$$

This is just the result of Givoli and Keller [1], which they introduced to eliminate a large circular region from a computational domain, thereby reducing the size of the computational domain.

Second, suppose that the scatterer is small compared to the wavelength $2\pi/k$. Then only the element S_{00} of the scattering matrix S is significantly different from zero. Consequently (2.9) yields $T_{00} = J_0(kR) + H_0^{(1)}(kR)\, S_{00}$ while all the other T_{nj} are given by $T_{nj} = J_n(kR)\, \delta_{nj}$, $(n, j) \neq (0, 0)$. Again, T is diagonal and T^{-1} exists if and only if each diagonal element of T is non-zero. When this is the case, (2.13) simplifies to

$$\begin{aligned} \partial_r u(R, \theta) &= \frac{k}{2\pi} \cdot \frac{J_0'(kR) + H_0^{(1)\prime}(kR)\, S_{00}}{J_0(kR) + H_0^{(1)}(kR)\, S_{00}} \cdot \int_0^{2\pi} u(R, \theta')\, d\theta' \\ &\quad + \frac{k}{2\pi} \sum_{n \neq 0} \frac{J_n'(kR)}{J_n(kR)} \int_0^{2\pi} e^{-in\theta'} u(R, \theta')\, d\theta' \cdot e^{in\theta}. \end{aligned} \tag{3.2}$$

When $S_{00} = 0$ this reduces to (3.1).

Third, we suppose that the scatterer is axially symmetric. Then S is diagonal so that $S_{nj} = S_{nn}\delta_{nj}$ and therefore $T_{nj} = [J_n(kR) + H_n^{(1)}(kR)\, S_{nn}]\, \delta_{nj}$. If every diagonal entry T_{nn} is not zero, T is invertible and T^{-1} is diagonal. Then (2.13) becomes

$$\partial_r u(R, \theta) = \frac{k}{2\pi} \sum_{n=-\infty}^{\infty} \frac{J_n'(kR) + H_n^{(1)\prime}(kR)\, S_{nn}}{J_n(kR) + H_n^{(1)}(kR)\, S_{nn}} \times \int_0^{2\pi} e^{-in\theta'} u(R, \theta')\, d\theta' \cdot e^{in\theta}. \tag{3.3}$$

This reduces to (3.2) when $S_{nn} = 0$ for $n \neq 0$ and to (3.1) when all $S_{nn} = 0$.

The scattering matrix S must be determined separately by analytical, numerical, or experimental means. For example, suppose that D is a circular disk of radius a with $u_r(a, \theta) = 0$. Then (2.5) shows that $\partial_r u_n(a) = 0$ and (2.6) yields

$$\frac{b_n}{a_n} = -\frac{J_n'(ka)}{H_n^{(1)\prime}(ka)} \equiv S_{nn}. \tag{3.4}$$

Thus in this special case S is diagonal and it can be found explicitly.

4. EQUIVALENCE OF THE REDUCED PROBLEM TO THE ORIGINAL PROBLEM

We shall now consider the reduced problem in the domain Ω/R. The boundary condition (2.13) is imposed at the inner boundary $r = R$. Our goal is to show that this problem has a unique solution which is exactly the same as the restriction of the solution of the original problem to this domain.

We begin by formulating the original problem as follows:

$$\Delta u + k^2 u = 0 \qquad (r, \theta) \in \Omega/D \tag{4.1}$$

$$Bu = 0 \qquad (r, \theta) \in \partial D \tag{4.2}$$

$$B_1 u = g \qquad (r, \theta) \in \partial\Omega. \tag{4.3}$$

In (4.3) g is a given or incident field and B_1 is some linear boundary operator. For instance, B_1 might be the operator in the exact non-reflecting boundary condition on $\partial\Omega$. In any case, we assume that the original problem (4.1)–(4.3) has a unique solution $u^0(r, \theta)$.

Next we consider the inner problem of solving (4.1) in R/D with the boundary condition (4.2) on the inner boundary ∂D and the outer boundary condition

$$u = U(\theta), \qquad r = R. \tag{4.4}$$

Here $U(\theta) = u^0(R, \theta)$ is the solution of (4.1)–(4.3) evaluated on $r = R$. Our second hypothesis is that this inner problem—(4.1) in R/D, (4.2), and (4.4)—has a unique solution $u^i(r\theta)$. This hypothesis implies that k is not an eigenvalue of the inner problem. Then from the solution of the inner problem we compute $u_r^i(R, \theta)$ and it is a linear functional of $U(\theta) = u^i(R, \theta)$, which we write in the form (2.3)

$$u_r(R, \theta) = Mu(R, \theta). \tag{4.5}$$

The operator M is given explicitly by (2.13).

Now we shall prove the following theorem:

THEOREM. *Suppose that the original problem* (4.1)–(4.3) *has a unique solution* $u^0(r, \theta)$ *and that the inner problem* (4.1) *in* R/D, (4.2), *and* (4.4) *also has a unique solution for any* $U(\theta)$. *Then the reduced problem* (4.1) *in* Ω/R, (4.3), *and* (4.5) *has a unique solution and it is equal to the restriction of* $u^0(r, \theta)$ *to* Ω/R.

Proof. The solution $u^0(r, \theta)$, when restricted to R/D, is a solution of the inner problem. By the hypothesis it is therefore the unique solution of that problem, so it satisfies (4.5). Since u^0 also satisfies (4.1) in Ω/R and (4.3), its restriction to Ω/R is a solution of the reduced problem. Therefore it remains to be proved that this is the only solution to the reduced problem.

Let us suppose that there were two solutions u^r and $\tilde{u}^r$ of the reduced problem which have two different values $U(\theta)$ and $\tilde{U}(\theta)$ on $r = R$. Then by using U and $\tilde{U}$ in (4.4) we would obtain two solutions u^i and $\tilde{u}^i$ of the inner problem. Now (4.5) is satisfied by u^r and $\tilde{u}^r$ as well as by u^i and $\tilde{u}^i$. Thus we have $u^r(R, \theta) = U(\theta) = u^i(R, \theta)$ and $u_r^r(R, \theta) = MU(\theta) = u_r^i(R, \theta)$ and, similarly, $\tilde{u}^r = \tilde{u}^i$ and $\tilde{u}_r^r = \tilde{u}_r^i$ on $r = R$. Therefore the pair u^r in Ω/R and u^i in R/D yield a function u in Ω/D which is continuous with continuous first derivatives. Similarly, $\tilde{u}^r$ in Ω/R and $\tilde{u}^i$ in R/D yield a C^1 function $\tilde{u}$ in Ω/D. Both the functions u and $\tilde{u}$ so defined are solutions of the original problem, so by the uniqueness hypothesis they are identical. Therefore $u^r = \tilde{u}^r$, so the reduced problem has only one solution. This completes the proof of the theorem.

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